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Sedimentation in a dilute dispersion of spheres

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The dispersion considered consists of a large number of identical small rigid spheres with random positions which are falling through Newtonian fluid under gravity. The volume fraction of the spheres (c) is small compared with unity. The dispersion is statistically homogeneous, and the axes of reference are chosen so that the mean volume flux across any stationary surface is zero. The problem is to determine the mean value of the velocity of a sphere ($\mathbf{U}$). In § 3 there is described a systematic and rigorous procedure which overcomes the familiar difficulty presented by the occurrence of divergent integrals, essentially by the choice of a quantity $\mathbf{V}$ whose mean value can be found exactly and which has the same long-range dependence on the position of a second sphere as $\mathbf{U}$ so that the mean of $\mathbf{U}-\mathbf{V}$ can be expressed in terms of an absolutely convergent integral. The result is that, correct to order c, the mean value of $\mathbf{U}$ is $\mathbf{U}_0(1-6{\cdot}55\,c)$, where $\mathbf{U}_0$ is the velocity of a single sphere in unbounded fluid. The only assumption made in the calculation is that the centres of spheres in the dispersion take with equal probability all positions such that no two spheres overlap; arguments are given in support of this assumption, which is expected to be valid only when the spheres are identical. Calculations which assume a simple regular arrangement of the spheres or which adopt a cell model of the hydrodynamic interactions give the quite different result that the change in the mean speed of fall is proportional to $c^{\frac{1}{3}}$, for reasons which are made clear.

The general procedure described here is expected to be applicable to other problems concerned with the effect of particle interactions on the average properties of dispersions with small volume fraction of the particles.

1. Introduction

When a homogeneous mixture of solid particles and a fluid is allowed to stand in a container, the particles settle out under gravity at a rate which depends on their size, shape, excess weight and concentration. The dependence on concentration arises from the interaction between particles, exerted by means of the velocity distribution generated in the fluid surrounding each moving particle. The effects of size and shape of the particles take the simplest possible form when the particles are identical rigid spheres of such small size that the Reynolds number of the fluid motion is small and inertia forces can be neglected. The mean speed of fall of a particle is then proportional to the excess weight and is otherwise a function primarily of the volume fraction of the particles. Some authors have maintained that the shape of the container walls is also relevant, but this is intrinsically unlikely, at any rate in a case in which the vessel contains a large number of particles uniformly dispersed throughout the fluid. The settling speed is found to be less than for an isolated particle (in the absence of particle agglomeration, of the kind reported by Kaye & Boardman 1962), and the phenomenon is often referred to as 'hindered settling'.

In the case of a cloud of particles which is surrounded by clear fluid, instead of being bounded by vessel walls, the motion of the particles depends also on the dimensions and shape of the cloud, like a finite-sized blob of one fluid falling through a second and less dense fluid. This presents a different problem, usually one in which the velocity of the cloud as a whole becomes so large that inertia forces are significant. The essential difference between the two problems lies not so much in the presence or absence of rigid boundaries as in the spatial variation of the statistical properties of the dispersion. We assume here that the distribution of particles in the ambient fluid is statistically homogeneous, and that the linear dimensions of the outer boundary of the fluid are large relative to the average distance between particles. As a consequence, no spatial variation of the mean velocity in the dispersion can be generated by gravity.

It is convenient in theoretical work to use axes of reference such that the mean velocity at a point in the dispersion (or, equivalently, the mean flux of material volume per unit area across any small stationary plane surface in the dispersion) is zero, although there are some practical contexts, such as a fluidized bed of particles, in which different axes are more natural.

Although there have been many contributions to the problem of determining the effect of concentration on the settling speed of small rigid spheres, beginning with Smoluchowski (1912), it has remained a challenge, even for the case of a dilute dispersion for which the volume fraction of the particles is small compared with unity. Many different theoretical and empirical formulae have been advanced, but not one of the available theoretical arguments is wholly satisfactory. The available observations appear not to be sufficiently accurate or consistent to specify the relationship closely. An account of past work on the problem is to be found in chapter 8 of the book by Happel & Brenner (1965).

The difficulty in the determination of the hydrodynamic interaction of the particles derives from the slowness with which the velocity disturbance in the fluid due to an isolated falling particle decreases to zero at increasing distance and, to a lesser extent, from the random arrangement of the particles in a real dispersion. The magnitude of the fluid velocity at distance r from a single sphere of radius a falling with speed U_0 varies asymptotically as $U_0 a/r$, and so a straightforward attempt to sum the contributions to the velocity at one point from an indefinitely large number of falling spheres in a homogeneous dispersion leads to a series or an integral which diverges strongly. The main objective of work on the problem has been to overcome this obstacle.

Previous theoretical investigations fall into three groups, corresponding to the assumptions made about the arrangement of the spherical particles in the dispersion and the nature of their interaction. In the first group are calculations which suppose, for mathematical convenience, that the centres of the spheres lie in some regular geometrical pattern, such as a cubical array, with the length scale of the array being of order $ac^{-\frac{1}{3}}$, where a is the radius of each of the spheres and c is the fraction of the total volume occupied by the spheres. It could not be maintained that in a real dispersion the particles are arranged regularly, and the implicit hypothesis underlying these investigations is presumably that the dependence of fall speed on concentration is much the same for the assumed regular arrangement of particles as it is for the disordered arrangement found in practice. These calculations yield the result that for a dilute dispersion ($c \ll 1$) the fractional reduction in the fall speed due to particle interactions is proportional to $c^{\frac{1}{3}}$, with a constant of proportionality which is of order unity and which varies with the type of arrangement assumed (simple cubic, body-centred cubic, rhombohedral, etc.). Hasimoto (1959) recognized that the way to overcome the difficulty of summing contributions from the various spheres in a regular array is to represent the local velocity as a Fourier series (which ensures the homogeneity of the dispersion) and to solve the equation for the Fourier coefficients subject to the boundary conditions corresponding to forces applied to the fluid at the positions of the spheres.

In the second group are calculations which use a 'cell' model of the interaction effects. The assumption here is that the average hydrodynamic effect on one sphere of the presence of all the other spheres in the dispersion is equivalent to that of a boundary, usually taken as spherical, enclosing the sphere under consideration. The radius of this outer spherical boundary is usually chosen as $ac^{-\frac{1}{3}}$, where a is the radius of the rigid sphere, thereby making the fraction of volume occupied by solid material the same in the cell as in the real dispersion. The motion of the fluid in the cell satisfies the no-slip condition at the surface of the rigid sphere, which is falling under gravity, and some suitably chosen condition at the stationary artificial outer boundary of the cell. One simple choice is that the fluid velocity is zero there; this and several alternative outer boundary conditions have been adopted by different authors. All these calculations with a cell radius proportional to $c^{-\frac{1}{3}}$ give a fractional reduction in the fall speed which is proportional to $c^{\frac{1}{3}}$ for $c \ll 1$. The constant of proportionality is again of order unity, but is not the same as that found for a regular array of spheres.

Investigators in the third group have used statistical analytical methods in an attempt to determine the hindered settling of a random distribution of spheres in a dilute dispersion (Burgers 1942; Pyun & Fixman 1964). Burgers tried a variety of ways of overcoming the difficulty presented by the lack of absolute convergence of the sum of the separate effects of an indefinitely large number of falling spheres on a given sphere, both for a random distribution and for a regular arrangement, and his sequence of papers is remarkable for the number of different answers provided. Burgers recognized the arbitrariness of some of his summation procedures, and was uncertain whether he had found expressions for the change in fall speed due to particle interactions which were independent of the shape of the container walls. I believe, nevertheless, that his papers do contain essentially the right approach, as well as some wrong steps, and that he was nearer to the correct answer than he or later writers have supposed. Pyun & Fixman employed a generally similar statistical approach, and improved on

Burgers's calculation in one detailed respect but erred in not following Burgers in another respect. For a random distribution of spheres Burgers and Pyun & Fixman found that the average change in the fall speed is proportional to c, a result which is not generally accepted in the literature, perhaps because the statistical methods used to obtain it were neither clear nor convincing, and perhaps because it is so different in form from that found either for a regular array of spheres or from the cell model.

In this paper we shall take it for granted that in a dispersion containing a large number of particles the arrangement of the particles in the ambient fluid is disordered and that only a statistical description of the particle locations is significant. It will be shown by rigorous methods that the change in the mean settling speed due to particle interactions in a dilute dispersion of rigid spheres is proportional to c, and the constant of proportionality will be determined.

There are some common features of the present problem of determining the velocity of sedimentation in a dilute dispersion correct to the order c and the problem of finding one of the bulk transport properties of a dilute dispersion correct to the order c^2, where in both cases c is the volume fraction of the phase present in the form of discrete particles. Included among these transport properties are the effective thermal conductivity of a stationary dispersion, the effective viscosity of a suspension of neutrally buoyant particles in simple shearing motion, and the effective elastic shear modulus for a dispersion of one solid material in another. In all these cases it is necessary to take into account the interaction of different particles, and in all these cases the straight-forward process of summing the separate effects of each of many particles on a given particle is frustrated by failure of the sums to converge absolutely. The general method that has been devised to overcome the difficulties of the present sedimentation problem is expected to be applicable also to these other similar problems.

2. Formulation of the problem

We consider a statistically homogeneous dispersion of identical rigid spherical particles of radius a in a Newtonian ambient fluid of viscosity μ. The flux of volume of material (which may be either fluid or solid) per unit area across any stationary plane surface in the dispersion defines a local velocity vector whose mean is uniform, and the axes of reference are chosen so that this mean velocity is zero. Inertia forces on either the solid particles or the fluid will be neglected.

The translational velocity of one particular sphere at one instant in one realization of the dispersion is $\mathbf{U}$ say. If this sphere were alone in infinite fluid and falling under gravity its velocity would be

$$\mathbf{U}_0 = \frac{2a^2(\rho_s - \rho)}{9\mu}\mathbf{g}, \tag{2.1}$$

where ρ_s is the density of the solid particle and ρ is that of the fluid. The velocity $\mathbf{U}$ of a particular sphere differs from $\mathbf{U}_0$ owing to the hydrodynamic interaction between the various particles in the dispersion, and $\mathbf{U} - \mathbf{U}_0$ is a random quantity with a non-zero mean which depends on the concentration of the particles by volume and whose value we wish to determine.

The velocity and pressure distributions in the fluid are governed by the linear Stokes equations

$$\mu\nabla^2\mathbf{u} = \nabla p, \quad \nabla.\mathbf{u} = 0. \tag{2.2}$$

The motion of a particular sphere is subject to the conditions that the resultant force exerted on it by the fluid is $-\frac{4}{3}\pi a^3(\rho_s-\rho)\mathbf{g}$ and the resultant couple about its centre is zero; and the no-slip condition must be satisfied at the surface of the sphere. It is known that in these circumstances the whole flow field within a region V is determined by the instantaneous positions of the spheres within V, together with the value of either the velocity or the force exerted per unit area at each point of the boundary of V. And in the case of a homogeneous dispersion the velocity at any point in V is evidently independent of the size and shape of the chosen region and of the conditions at the boundary of V, provided (1) that the region is large enough to contain many spheres, and (2) that the conditions at the boundary of V are compatible with zero mean velocity everywhere in V. We may thus suppose that the velocity and other field variables at a point $\mathbf{x}$ in the dispersion, in one realization, are determined by the instantaneous positions of the centres of the N spheres in a region of volume V containing $\mathbf{x}$, where $N \gg 1$. It follows that an average of some property of the dispersion at $\mathbf{x}$ over the ensemble of realizations is effectively an average over the ensemble of sphere positions relative to $\mathbf{x}$.

We shall denote the set of position vectors of the centres of N spheres in one realization by $\mathscr{C}_N$, and term it a configuration of N spheres. The probability density of the configuration is $P(\mathscr{C}_N)$, meaning that the probability of sphere centres being located simultaneously in the volume elements $\delta\mathbf{r}_1, \delta\mathbf{r}_2, \ldots, \delta\mathbf{r}_N$ about the points

$$\mathbf{x}+\mathbf{r}_1, \mathbf{x}+\mathbf{r}_2, \ldots, \mathbf{x}+\mathbf{r}_N$$

is

$$P(\mathscr{C}_N)\,\delta\mathscr{C}_N = P(\mathbf{x}+\mathbf{r}_1, \mathbf{x}+\mathbf{r}_2, \ldots, \mathbf{x}+\mathbf{r}_N)\,\delta\mathbf{r}_1, \delta\mathbf{r}_2, \ldots, \delta\mathbf{r}_N.$$

The position vector $\mathbf{x}$ here specifies a reference point for the configuration $\mathscr{C}_N$; $P(\mathscr{C}_N)$ is of course independent of $\mathbf{x}$ in view of the homogeneity of the dispersion, and we need show the reference point only when configurations are being described explicitly. The N spheres in the region of volume V are identical and so we have the normalization relation

$$\int P(\mathscr{C}_N)\,d\mathscr{C}_N = N!, \tag{2.3}$$

where here and later it is understood that each of the N volume integrals comprising integration with respect to $\mathscr{C}_N$ is taken over the whole of the volume V. The probability density of the position vector of a single sphere centre will be written as

$$P(\mathscr{C}_1) = P(\mathbf{x}+\mathbf{r}) = N/V = n, \tag{2.4}$$

where n is the average number of spheres per unit volume.

We also introduce the *conditional* probability density $P(\mathscr{C}_N|\mathbf{x})$ which refers to a dispersion in which there are $N+1$ sphere centres in V; $P(\mathscr{C}_N|\mathbf{x})\,\delta\mathscr{C}_N$ is the probability of a configuration of N sphere centres being found in the range $\delta\mathscr{C}_N$

about $\mathscr{C}_N$, given that there is a sphere centre at the point $\mathbf{x}$. The relation (2.3) also holds when $P(\mathscr{C}_N)$ is replaced by $P(\mathscr{C}_N|\mathbf{x})$. A connexion between the conditional and unconditional probabilities is provided by the identity

$$P(\mathscr{C}_N) = P(\mathbf{x}+\mathbf{r}_k)\,P(\mathscr{C}_{N-1}|\mathbf{x}+\mathbf{r}_k) = nP(\mathscr{C}_{N-1}|\mathbf{x}+\mathbf{r}_k), \qquad (2.5)$$

where $\mathbf{x}+\mathbf{r}_k$ is the location of one of the sphere centres of $\mathscr{C}_N$. Likewise, with an obvious notation,

$$P(\mathscr{C}_N|\mathbf{x}) = P(\mathbf{x}+\mathbf{r}_k|\mathbf{x})\,P(\mathscr{C}_{N-1}|\mathbf{x},\mathbf{x}+\mathbf{r}_k). \qquad (2.6)$$

It will be assumed that there is no long-range order in the dispersion, and that the probabilities of sphere centres being at points whose separation is large compared with a sphere radius are independent. In particular we have

$$P(\mathscr{C}_N|\mathbf{x}) \approx P(\mathscr{C}_N) \qquad (2.7)$$

when each of the points of $\mathscr{C}_N$ is at a distance from $\mathbf{x}$ which is large compared with a.

We may now express the average of some quantity $G(\mathbf{x},\mathscr{C}_N)$ which is associated with a point $\mathbf{x}$ in the dispersion (G necessarily being defined for points both in the fluid and in the rigid spheres) and which is determined by the configuration of N spheres, as

$$\bar{G} = \frac{1}{N!}\int G(\mathbf{x},\mathscr{C}_N)\,P(\mathscr{C}_N)\,d\mathscr{C}_N. \qquad (2.8)$$

We shall also be concerned with the average of a quantity, $H(\mathbf{x},\mathscr{C}_N)$ say, which is associated with a sphere with centre at the point $\mathbf{x}$; this requires the conditional probability density and is

$$\bar{H} = \frac{1}{N!}\int H(\mathbf{x},\mathscr{C}_N)\,P(\mathscr{C}_N|\mathbf{x})\,d\mathscr{C}_N. \qquad (2.9)$$

The interpretation of the expressions (2.8) and (2.9) is that $G(\mathbf{x},\mathscr{C}_N)$ and $H(\mathbf{x},\mathscr{C}_N)$ in the integrands represent the values taken by the quantity concerned at the point $\mathbf{x}$ when the configuration of the centres of the surrounding spheres has the form $\mathscr{C}_N$.

Our objective is to calculate the mean value of the velocity of fall of a sphere, which is given by (2.9) with $\mathbf{U}$ in place of H. We shall thus need to know both the configurational statistics for a group of spheres and the dependence of the velocity of one sphere on the relative locations of surrounding spheres.

We have yet to make use of the assumption that the dispersion is dilute, with an average spacing of sphere centres ($n^{-\frac{1}{3}}$) which is large compared with a sphere radius. The probability that just one of the sphere centres of the configuration $\mathscr{C}_N$ is located at a distance from the reference point $\mathbf{x}$ which is a multiple of a of order of magnitude unity is of order c, where

$$c = \tfrac{4}{3}\pi a^3 n$$

is the volume fraction of the spheres in the dispersion, and so is a small quantity. Likewise the probability that two sphere centres are simultaneously at such a distance from $\mathbf{x}$ is of order c^2. It follows that, in any case in which the quantity G decreases to zero sufficiently rapidly with increase of the distance r of a single rigid sphere from the point $\mathbf{x}$, the average (2.8) may be calculated, with an error of order c^2, as if the surrounding configuration contained just one sphere; 'sufficiently rapidly' evidently means as rapidly as $(a/r)^{3+\epsilon}$, where $\epsilon > 0$, in order to ensure absolute convergence of an integral over all positions of one sphere centre of the configuration with uniform probability density of location. This approximation corresponds to using the identity (2.5) and the approximate relation (2.7) to replace $P(\mathscr{C}_N)$ in the integrand of (2.8) by

$$P(\mathbf{x}+\mathbf{r}_k)\,P(\mathscr{C}_{N-1})$$

for that part of the range of integration of $\mathbf{r}_k$ for which r_k/a (where $r_k = |\mathbf{r}_k|$) is of order unity and to ignoring the influence of other spheres on the value of G. Thus we have

$$\begin{aligned}\bar{G} &= \sum_{k=1}^{N} \frac{1}{N!}\int P(\mathscr{C}_{N-1})\,d\mathscr{C}_{N-1}\int G(\mathbf{x},\mathbf{x}+\mathbf{r}_k)\,P(\mathbf{x}+\mathbf{r}_k)\,d\mathbf{r}_k + O(c^2)\\ &= \int G(\mathbf{x},\mathbf{x}+\mathbf{r})\,P(\mathbf{x}+\mathbf{r})\,d\mathbf{r} + O(c^2),\end{aligned} \tag{2.10}$$

where $G(\mathbf{x},\mathbf{x}+\mathbf{r})$ stands for the value of the function G at the point $\mathbf{x}$ in the presence of a single sphere with centre at $\mathbf{x}+\mathbf{r}$.

Exactly the same remarks may be made about any quantity $H(\mathbf{x},\mathscr{C}_N)$ associated with a sphere with centre at $\mathbf{x}$ which is significantly different from zero only when at least one sphere of the configuration $\mathscr{C}_N$ is within a distance of order a from $\mathbf{x}$, with the conditional probability $P(\mathbf{x}+\mathbf{r}|\mathbf{x})$ replacing $P(\mathbf{x}+\mathbf{r})$ in the counterpart of (2.10).

However, it must be noted that not all the quantities represented by G or H occurring in the problem under discussion satisfy the required condition of decreasing to zero sufficiently rapidly with increase of the distance of a single rigid sphere from the point $\mathbf{x}$. In particular, the velocity at the point $\mathbf{x}$ due to one falling sphere with centre at $\mathbf{x}+\mathbf{r}$ behaves as r^{-1} when $r/a \gg 1$, and so it is definitely not permissible in this case to disregard contributions to an average of the form (2.8) from spheres at distances large compared with a; and the approximation (2.10) for the average would be a divergent integral if $G(\mathbf{x},\mathbf{x}+\mathbf{r})$ were to be interpreted as the velocity at $\mathbf{x}$ due to a sphere with centre at $\mathbf{x}+\mathbf{r}$. On the other hand, without a reduction of an average to an expression involving only a small number of spheres it does not seem to be possible to make progress analytically. This is the central difficulty of the problem to which we have already referred.

3. The method of solution

Our objective is to determine the mean velocity of a sphere, which is represented formally by

$$\bar{\mathbf{U}} = \frac{1}{N!}\int \mathbf{U}(\mathbf{x}_0,\mathscr{C}_N)\,P(\mathscr{C}_N|\mathbf{x}_0)\,d\mathscr{C}_N. \tag{3.1}$$

The sphere with centre at $\mathbf{x}_0$ whose velocity is $\mathbf{U}(\mathbf{x}_0,\mathscr{C}_N)$ will be termed the test sphere. We seek in particular an approximation to $\bar{\mathbf{U}}$ which is correct to the order c. Calculations of the flow fields due to one or two spheres falling in infinite fluid are feasible, whereas they are not for a larger number of spheres, and so we wish to reduce (3.1) in some way to a consideration of a group of no more than two spheres. The straight-forward idea of regarding the integral in (3.1) as effectively an integration over the location of just one sphere in the configuration $\mathscr{C}_N$, on the grounds that the chance of two spheres being simultaneously close enough to $\mathbf{x}_0$ to influence the velocity of the test sphere is of order c^2 and so negligible, is invalidated by the slowness of the decrease to zero of the influence of one falling sphere on $\mathbf{U}(\mathbf{x}_0)$ with increasing distance from $\mathbf{x}_0$.

The procedure to be adopted here is to look for a quantity whose mean is known *exactly* from some overall condition or constraint in the specification of the problem and whose value at $\mathbf{x}_0$ has the *same* long-range dependence on the presence of a sphere at $\mathbf{x}_0+\mathbf{r}$ as the velocity of the test sphere; and, once found, the *difference* between $\overline{\mathbf{U}}$ and the mean of this quantity can be expressed as an integral like (3.1) which can then legitimately be reduced to an integration over the location of just one sphere in the configuration $\mathscr{C}_N$ and evaluated explicitly.

The choice of the desired quantity becomes evident as soon as we investigate the asymptotic form of the dependence of $\mathbf{U}(\mathbf{x}_0,\mathscr{C}_N)$ on the configuration $\mathscr{C}_N$ as the distance of the spheres in $\mathscr{C}_N$ that are nearest to $\mathbf{x}_0$ becomes large. Now when all the spheres of $\mathscr{C}_N$ are well away from the point $\mathbf{x}_0$, the test sphere may be regarded as immersed in fluid which in the absence of that sphere would have approximately uniform velocity over a region of linear dimensions $2a$; and $\mathbf{U}(\mathbf{x}_0,\mathscr{C}_N)$ would then be approximately equal to the sum of $\mathbf{U}_0$ and that uniform velocity. This suggests we should look closely at the relation between $\mathbf{U}(\mathbf{x}_0,\mathscr{C}_N)$ and the velocity distribution that would exist in the dispersion if the test sphere were replaced by fluid of viscosity μ without change of the configuration $\mathscr{C}_N$. This latter velocity, whose value at point $\mathbf{x}$ is denoted by $\mathbf{u}(\mathbf{x},\mathscr{C}_N)$ (the configuration $\mathscr{C}_N$ here being specified relative to $\mathbf{x}_0$ still), will in general be non-uniform on the spherical surface A_0 centred on $\mathbf{x}_0$ with radius a.

In order to determine the sphere velocity $\mathbf{U}(\mathbf{x}_0,\mathscr{C}_N)$ in terms of $\mathbf{u}(\mathbf{x},\mathscr{C}_N)$ we need to find a surface distribution of forces, or Stokeslets, on A_0 whose vector resultant is $\frac{4}{3}\pi a^3(\rho_s-\rho)\,\mathbf{g}$ and which, if applied to the fluid in the presence of all the other rigid freely moving spheres in the dispersion, would generate a fluid velocity on A_0 which is the sum of a uniform vector and the variable quantity $-\mathbf{u}(\mathbf{x})$. It is shown in the appendix that, if the need to continue to satisfy the no-slip condition at the surface of spheres other than the test sphere is disregarded for the moment, the unique translational velocity of the test sphere is exactly $\mathbf{U}_0+\mathbf{V}$, where

$$\mathbf{V}(\mathbf{x}_0,\mathscr{C}_N) = \frac{1}{4\pi a^2}\int_{A_0}\mathbf{u}(\mathbf{x},\mathscr{C}_N)\,dA.$$

On representing $\mathbf{u}(\mathbf{x})$ as a Taylor series in $\mathbf{x}-\mathbf{x}_0$ and integrating over A_0, and using the slow-motion equation $\nabla^4\mathbf{u} = 0$, we see that

$$\mathbf{V}(\mathbf{x}_0,\mathscr{C}_N) = \mathbf{u}(\mathbf{x}_0,\mathscr{C}_N)+\tfrac{1}{6}a^2\{\nabla^2\mathbf{u}(\mathbf{x},\mathscr{C}_N)\}_{\mathbf{x}=\mathbf{x}_0}; \tag{3.2}$$

this expression for the translational velocity of a rigid sphere due to a non-uniform environment was obtained many years ago by Faxen (see Oseen 1927).

The expression (3.2) takes account of the velocity distribution in the fluid near $\mathbf{x}_0$ due to the motion of all spheres other than the test sphere, but it is incomplete because the forces acting at the surface of the test sphere need to be accompanied by image systems in the spherical boundaries of all other spheres in order to ensure that the no-slip condition continues to be satisfied at those boundaries. The effect of the presence of the test sphere is to induce at distance r in the surrounding fluid a velocity whose order of magnitude is $U_0 a/r$, a velocity gradient of order $U_0 a/r^2$, etc. A rigid sphere of radius a whose centre is at distance r from that of the test sphere will as a consequence acquire an additional translational velocity (with no associated change in the stress distribution at its surface), and in addition there will be changes in the stress distribution at the sphere surface due to the ambient velocity gradient which have a net force dipole magnitude of order $\mu U_0 a^4/r^2$. This new force dipole will in turn induce a change in the velocity distribution near the test sphere, and in particular the test sphere will be given an additional translational velocity of order $U_0 a^4/r^4$. All other spheres in the dispersion will have a similar effect on the test sphere. The translational velocity of the test sphere with centre at $\mathbf{x}_0$ should thus be written as

$$\mathbf{U}(\mathbf{x}_0, \mathscr{C}_N) = \mathbf{U}_0 + \mathbf{V}(\mathbf{x}_0, \mathscr{C}_N) + \mathbf{W}(\mathbf{x}_0, \mathscr{C}_N), \tag{3.3}$$

where $\mathbf{W}$ represents the effect of the image systems, in the boundaries of all other spheres, of the Stokeslets at the surface of the test sphere. Note (1) that these Stokeslets include both those related to the gravity force on the test sphere and those required to cancel the velocity field $\mathbf{u}$, (2) that the contribution to $\mathbf{W}$ from one other sphere at distance r varies asymptotically as r^{-4}, showing that the sum of the separate contributions from an indefinitely large number of spheres located with uniform probability density would converge, and (3) that $\mathbf{W}$ is defined precisely as the difference between $\mathbf{U}-\mathbf{U}_0$ and the velocity $\mathbf{V}$ given by (3.2).

With the relations (3.2) and (3.3) in mind it is possible to see how to overcome the difficulty referred to at the beginning of this section. In place of $\overline{\mathbf{U}}$ we now evaluate the two mean quantities $\overline{\mathbf{V}}$ and $\overline{\mathbf{W}}$. For the latter we may use the kind of approximation represented by (2.10), because the value of $\mathbf{W}$ due to one sphere of the configuration at distance r from $\mathbf{x}_0$ decreases so rapidly, as $r/a \to \infty$, that only those spheres of the configuration $\mathscr{C}_N$ which are within a distance from $\mathbf{x}_0$ of order a have a significant effect. Thus we have

$$\begin{aligned}\overline{\mathbf{W}} &= \frac{1}{N!}\int \mathbf{W}(\mathbf{x}_0, \mathscr{C}_N)\, P(\mathscr{C}_N|\mathbf{x}_0)\, d\mathscr{C}_N \\ &= \int \mathbf{W}(\mathbf{x}_0, \mathbf{x}_0+\mathbf{r})\, P(\mathbf{x}_0+\mathbf{r}|\mathbf{x}_0)\, d\mathbf{r} + O(c^2),\end{aligned} \tag{3.4}$$

where

$$\mathbf{W}(\mathbf{x}_0, \mathbf{x}_0+\mathbf{r}) = \mathbf{U}(\mathbf{x}_0, \mathbf{x}_0+\mathbf{r}) - \mathbf{U}_0 - \mathbf{u}(\mathbf{x}_0, \mathbf{x}_0+\mathbf{r}) - \tfrac{1}{6}a^2\{\nabla^2\mathbf{u}(\mathbf{x}, \mathbf{x}_0+\mathbf{r})\}_{\mathbf{x}=\mathbf{x}_0}, \tag{3.5}$$

$\mathbf{U}(\mathbf{x}_0, \mathbf{x}_0+\mathbf{r})$ is the velocity of a sphere with centre at $\mathbf{x}_0$ in the presence of a second sphere with centre at $\mathbf{x}_0+\mathbf{r}$, and $\mathbf{u}(\mathbf{x}, \mathbf{x}_0+\mathbf{r})$ is the velocity at the point $\mathbf{x}$ in the presence of a sphere with centre at $\mathbf{x}_0+\mathbf{r}$.

All the difficulties presented by non-convergence are incorporated in the contribution

$$\overline{\mathbf{V}} = \overline{\mathbf{V}'} + \overline{\mathbf{V}''}$$

where
$$\left.\begin{aligned}\overline{\mathbf{V}'} &= \frac{1}{N!}\int \mathbf{u}(\mathbf{x}_0,\mathscr{C}_N)\,P(\mathscr{C}_N|\mathbf{x}_0)\,d\mathscr{C}_N,\\ \overline{\mathbf{V}''} &= \frac{1}{N!}\int \tfrac{1}{6}a^2\{\nabla^2\mathbf{u}(\mathbf{x},\mathscr{C}_N)\}_{\mathbf{x}=\mathbf{x}_0}P(\mathscr{C}_N|\mathbf{x}_0)\,d\mathscr{C}_N.\end{aligned}\right\} \tag{3.6}$$

It is not possible to reduce *either* of these two integrals to an integral with respect to the position of only one sphere, because $|\mathbf{u}|$ behaves as a/r at distance r from one falling sphere and $a^2|\nabla^2\mathbf{u}|$ behaves as a^3/r^3, the latter giving a decrease to zero as $r/a \to \infty$ which is only just too slow for absolute convergence of such an integral. We therefore try to evaluate the integrals in (3.6) with the aid of known exact mean values involving all the spheres in the configuration $\mathscr{C}_N$.

One obviously relevant result is that the mean value of the velocity at a point in the dispersion is zero, which is expressed formally as

$$\bar{\mathbf{u}} = \frac{1}{N!}\int \mathbf{u}(\mathbf{x},\mathscr{C}_N)\,P(\mathscr{C}_N)\,d\mathscr{C}_N = 0, \tag{3.7}$$

where the local velocity $\mathbf{u}$ has its ordinary physical meaning in each of the fluid and solid parts of the dispersion. The contribution $\overline{\mathbf{V}'}$ may thus be written as

$$\overline{\mathbf{V}'} = \frac{1}{N!}\int \mathbf{u}(\mathbf{x}_0,\mathscr{C}_N)\,\{P(\mathscr{C}_N|\mathbf{x}_0)-P(\mathscr{C}_N)\}\,d\mathscr{C}_N. \tag{3.8}$$

A reminder of the meaning of the symbols occurring in the integrand here may be in order. $\mathbf{u}(\mathbf{x}_0,\mathscr{C}_N)$ denotes the velocity at the point $\mathbf{x}_0$ in the presence of the configuration of N spheres represented by $\mathscr{C}_N$, and the two terms within curly brackets specify two probability distributions for $\mathscr{C}_N$; the first, $P(\mathscr{C}_N|\mathbf{x}_0)$, represents the distribution of $\mathscr{C}_N$ as it would be if the centre of another sphere was known to be at $\mathbf{x}_0$ (but note that this sphere is not present so far as the value of $\mathbf{u}(\mathbf{x}_0,\mathscr{C}_N)$ is concerned), and the second, $P(\mathscr{C}_N)$, is the unconditional distribution of $\mathscr{C}_N$, involving some configurations for which the point $\mathbf{x}_0$ lies within a rigid sphere.

The integral (3.8) is now in a form which allows reduction to an integral over the position of just one sphere of the configuration $\mathscr{C}_N$. The identities (2.5) and (2.6) enable us to rewrite the quantity within curly brackets in (3.8) as

$$P(\mathbf{x}_0+\mathbf{r}_k|\mathbf{x}_0)\,P(\mathscr{C}_{N-1}|\mathbf{x}_0,\mathbf{x}_0+\mathbf{r}_k)-P(\mathbf{x}_0+\mathbf{r}_k)\,P(\mathscr{C}_{N-1}|\mathbf{x}_0+\mathbf{r}_k),$$

where $\mathbf{x}_0+\mathbf{r}_k$ is one of the points of the configuration $\mathscr{C}_N$; and for those parts of the range of integration with respect to $\mathscr{C}_N$ in (3.8) for which r_k/a is of order unity and $r_l/a \gg 1$ for all l except $l = k$, this is approximately equal to

$$\{P(\mathbf{x}_0+\mathbf{r}_k|\mathbf{x}_0)-P(\mathbf{x}_0+\mathbf{r}_k)\}\,P(\mathscr{C}_{N-1}).$$

We may assume that the difference $P(\mathbf{x}_0+\mathbf{r}_k|\mathbf{x}_0)-P(\mathbf{x}_0+\mathbf{r}_k)$ tends to zero rapidly† as $r_k/a \to \infty$, and so we have

$$\overline{\mathbf{V}'} = \sum_{k=1}^{N}\frac{1}{N!}\iint \mathbf{u}(\mathbf{x}_0,\mathscr{C}_N)\,\{P(\mathbf{x}_0+\mathbf{r}_k|\mathbf{x}_0)-P(\mathbf{x}_0+\mathbf{r}_k)\}\,P(\mathscr{C}_{N-1})\,d\mathbf{r}_k\,d\mathscr{C}_{N-1}+O(c^2),$$

† We defer consideration of the actual expression for this quantity until the next section.

the error being of order c^2 since the probability of the neglected possibility that at least two spheres of $\mathscr{C}_N$ are located within a distance of order a from $\mathbf{x}_0$ is of this order. The leading approximation to $\mathbf{u}(\mathbf{x}_0, \mathscr{C}_N)$ when only the sphere centre at $\mathbf{x}_0 + \mathbf{r}_k$ is close to $\mathbf{x}_0$ is $\mathbf{u}(\mathbf{x}_0, \mathbf{x}_0 + \mathbf{r}_k)$ (and when no sphere of the configuration $\mathscr{C}_N$ is close to $\mathbf{x}_0$ the integrand is small in any event), whence

$$\overline{\mathbf{V}'} = \int \mathbf{u}(\mathbf{x}_0, \mathbf{x}_0 + \mathbf{r}) \{P(\mathbf{x}_0 + \mathbf{r} | \mathbf{x}_0) - P(\mathbf{x}_0 + \mathbf{r})\} \, d\mathbf{r} + O(c^2). \tag{3.9}$$

In order to evaluate the contribution $\overline{\mathbf{V}''}$ we introduce the deviatoric stress tensor

$$d_{ij} = \sigma_{ij} - \tfrac{1}{3}\delta_{ij}\sigma_{kk},$$

which is defined in both the fluid and solid parts of the dispersion and has the Newtonian form $2\mu e_{ij}$ in the fluid, where e_{ij} is the rate-of-strain tensor. Now $d_{ij}(\mathbf{x})$ is a stationary random function of position in a statistically homogeneous dispersion, and so has constant mean. It follows that the mean of $\partial d_{ij}/\partial x_j$ is zero, that is

$$\frac{1}{N!} \int \frac{\partial d_{ij}(\mathbf{x}, \mathscr{C}_N)}{\partial x_j} P(\mathscr{C}_N) \, d\mathscr{C}_N = 0, \tag{3.10}$$

in which the differentiation of d_{ij} with respect to $\mathbf{x}$ is carried out with $\mathscr{C}_N$ held fixed and the position vectors of the N sphere centres can be chosen as $\mathbf{x}_0 + \mathbf{r}_1$, $\mathbf{x}_0 + \mathbf{r}_2, \ldots, \mathbf{x}_0 + \mathbf{r}_N$. We shall divide the range of integration with respect to $\mathbf{r}_k$ into two parts, one of which is specified by $|\mathbf{x}_0 + \mathbf{r}_k - \mathbf{x}| \leqslant a$ (so that $\mathbf{x}$ lies inside or on the surface of the kth sphere) and makes a contribution to the integral in (3.10) which, with use of the identity (2.5), may be written as

$$\int \int_{|\mathbf{x}_0 + \mathbf{r}_k - \mathbf{x}| \leqslant a} \frac{\partial d_{ij}(\mathbf{x}, \mathscr{C}_N)}{\partial x_j} P(\mathscr{C}_{N-1} | \mathbf{x}_0 + \mathbf{r}_k) \, d\mathbf{r}_k \, d\mathscr{C}_{N-1}.$$

Now since the range of integration with respect to $\mathscr{C}_{N-1}$ here is unrestricted, it is permissible to carry out the integration with respect to $\mathbf{r}_k$ with the configuration $\mathscr{C}_{N-1}$ held fixed relative to the sphere centre $\mathbf{x}_0 + \mathbf{r}_k$, that is, with

$$P(\mathscr{C}_{N-1} | \mathbf{x}_0 + \mathbf{r}_k)$$

constant. The integral of $\partial d_{ij}/\partial x_j$ with respect to $\mathbf{r}_k$ over the volume of the kth sphere can then be transformed to an integral of d_{ij} over the surface of this sphere. A contribution of this kind is made by each of the N spheres separately, and on summing over all values of k, and noting that when $|\mathbf{x}_0 + \mathbf{r}_k - \mathbf{x}| > a$ for all k we may put

$$\frac{\partial d_{ij}}{\partial x_j} = \mu \nabla^2 u_i,$$

we have in place of (3.10)

$$\sum_{k=1}^{N} \frac{n}{N!} \int \mathbf{f}(\mathbf{x}_0 + \mathbf{r}_k, \mathscr{C}_{N-1}) \, P(\mathscr{C}_{N-1} | \mathbf{x}_0 + \mathbf{r}_k) \, d\mathscr{C}_{N-1}$$
$$+ \frac{1}{N!} \int_{|\mathbf{x}_0 + \mathbf{r}_k - \mathbf{x}| > a, \, \text{all } k} \mu \nabla^2 \mathbf{u}(\mathbf{x}, \mathscr{C}_N) \, P(\mathscr{C}_N) \, d\mathscr{C}_N = 0, \tag{3.11}$$

where $\mathbf{f}(\mathbf{x}_0+\mathbf{r}_k, \mathscr{C}_{N-1})$ is the total force exerted by the deviatoric stress on the surface of a rigid sphere with centre at $\mathbf{x}_0+\mathbf{r}_k$ in the presence of a configuration of $N-1$ other spheres. The first of these integrals is independent of $\mathbf{r}_k$, and so we can put $\mathbf{r}_k = 0$; and second is independent of $\mathbf{x}$, and we can put $\mathbf{x} = \mathbf{x}_0$.

The contribution $\overline{\mathbf{V}''}$ may now be written as

$$\overline{\mathbf{V}''} = \frac{1}{N!}\int_{r_k>a,\,\mathrm{all}\,k} \tfrac{1}{6}a^2\{\nabla^2\mathbf{u}(\mathbf{x},\mathscr{C}_N)\}_{\mathbf{x}=\mathbf{x}_0}\{P(\mathscr{C}_N|\mathbf{x}_0)-P(\mathscr{C}_N)\}\,d\mathscr{C}_N \\ -\frac{1}{(N-1)!}\int \frac{na^2}{6\mu}\mathbf{f}(\mathbf{x}_0,\mathscr{C}_{N-1})\,P(\mathscr{C}_{N-1}|\mathbf{x}_0)\,d\mathscr{C}_{N-1}; \quad (3.12)$$

it is permissible to impose the restriction on the range of integration for the integral coming from (3.6), since in any event $P(\mathscr{C}_N|\mathbf{x}_0) = 0$ for any r_k smaller than $2a$. The argument which led to the reduction of (3.8) to the approximate form (3.9) may also be applied to the first integral in (3.12). And for the second integral in (3.12) we note that the leading approximation to the mean value of $na^2\mathbf{f}/6\mu$ for $c \ll 1$ is $-\frac{1}{2}c\mathbf{U}_0$ (the drag due to viscous stress at the surface of a single falling sphere in infinite fluid being two-thirds of the total drag force), which is sufficiently accurate. The approximate expression for $\overline{\mathbf{V}''}$ is therefore

$$\overline{\mathbf{V}''} = \int_{r>a} \tfrac{1}{6}a^2\{\nabla^2\mathbf{u}(\mathbf{x},\mathbf{x}_0+\mathbf{r})\}_{\mathbf{x}=\mathbf{x}_0}\{P(\mathbf{x}_0+\mathbf{r}|\mathbf{x}_0)-P(\mathbf{x}_0+\mathbf{r})\}\,d\mathbf{r}+\tfrac{1}{2}c\mathbf{U}_0, \quad (3.13)$$

with an error of order c^2.

The final expression for the mean velocity of a sphere in the dispersion is

$$\overline{\mathbf{U}} = \mathbf{U}_0+\overline{\mathbf{V}'}+\overline{\mathbf{V}''}+\overline{\mathbf{W}}, \quad (3.14)$$

where $\overline{\mathbf{V}'}$ is given approximately by (3·9), $\overline{\mathbf{V}''}$ by (3.13), and $\overline{\mathbf{W}}$ by (3.4), in all three cases with an error of order c^2. The expressions for $\overline{\mathbf{V}'}$ and $\overline{\mathbf{V}''}$ and $\overline{\mathbf{W}}$ can be evaluated from a knowledge of the probability density of the location of one sphere relative to a second sphere in a statistically homogeneous dispersion and of the flow field due to two spheres falling through infinite fluid.

This completes the description of the general method of overcoming the difficulty caused by the slowness of the decrease of the velocity to zero with increasing distance from one falling sphere. The statistical aspects of the method have been set out in rather formal terms, partly because it is otherwise so difficult to avoid errors in a statistical argument and partly in order to facilitate the expected applications to other problems concerned with the mean properties of a mixture consisting of randomly dispersed particles in a uniform ambient medium.

4. The probability distribution of the separation of two spheres

It is evident that the integrals in (3.4), (3.9) and (3.13) are all of order cU_0; for the magnitude of $|\mathbf{u}|$ can be estimated as U_0a/r, that of both $P(\mathbf{x}_0+\mathbf{r}|\mathbf{x}_0)$ and $P(\mathbf{x}_0+\mathbf{r})$ by n, and the value of r/a at which the integrands become small is of order unity. Consequently, in order to obtain the mean sphere velocity correct to order c, we need determine only the leading approximation to $P(\mathbf{x}_0+\mathbf{r}|\mathbf{x}_0)$, that is, we need determine only the functional form corresponding to the limit $c \to 0$.

We now consider what statistical distribution of relative particle positions should be expected in a dispersion, taking account of both the initial conditions and the subsequent effect of hydrodynamic interactions. The initial conditions depend on the actual way in which the dispersion is obtained, and completely general statements cannot be made. However, it seems likely that any procedure such as violent shaking of the mixture in a closed container which causes strong fluctuating forces to act on the spheres will generate a distribution such that the locations of sphere centres have no statistical connexions other than those arising, either directly or indirectly, from the requirement that rigid spheres do not overlap.† This initial condition corresponds to production of the dispersion in a large volume V by the hypothetical process of choosing the locations of sphere centres one by one at random in V according to the rule that each new sphere centre can be placed with uniform probability anywhere in the accessible part of V (that is, at any point not closer than two sphere radii to any previously chosen point). Another way of generating such a distribution mathematically would be to suppose that the spheres exist in a vacuum and move in straight lines with constant velocity until they make elastic collisions at which momentum and energy are conserved, with the duration of this hypothetical motion being long enough for an equilibrium distribution to have been established.

The determination of the analytical form of the distribution of relative particle positions corresponding to this initial condition is a difficult problem in statistical geometry‡ for general values of the volume fraction c. However, in the limit $c \to 0$ the effect of the impenetrability of the spheres reduces to a simple exclusion of configurations for which overlappings of spheres occurs, with all allowed configurations being equally probable. Thus, for a dilute dispersion our supposed initial condition implies that the probability density of the location of one sphere centre relative to another is

$$P(\mathbf{x}+\mathbf{r}|\mathbf{x}) = \begin{Bmatrix} n & \text{if} & r \geqslant 2a \\ 0 & \text{if} & r < 2a \end{Bmatrix}, \tag{4.1}$$

where n is the uniform mean number density of spheres. Similar statements may be made about the probability densities of configurations of more than two spheres.

There is now the question of change of the initial statistical distribution of spheres due to the motion of the spheres under gravity. In most problems concerning dispersed spheres it is to be expected that hydrodynamic interaction will lead to some weighting of the different sphere configurations and ultimately to the establishment of a sphere distribution which is independent of the initial conditions. However, any such bias due to hydrodynamic interaction must be small in a dilute dispersion of identical spheres in which sedimentation has been going on for a

† In the case of very small particles there may exist electrostatic repulsive and intermolecular attractive forces between the particles which affect significantly the excluded volume associated with one particle, although we shall ignore any such effects here.

‡ Well known in the theory of the liquid state, see Fisher (1964, chap. 3) and Rice & Gray (1965, chap. 2).

limited time only. It is known that two identical spheres falling through infinite fluid maintain a constant relative position (as is evident from symmetry and reversibility), and so any tendency for certain configurations of neighbouring spheres to be preferred as a result of hydrodynamic action can arise only from triple or higher-order interactions. Now the fraction of all the spheres in one realization that instantaneously are located within a few radii of a second sphere is of order c, and this is also the fraction of the total time that a given sphere spends in proximity to a second sphere. Likewise the fraction of the total time that a given sphere spends in proximity to two other spheres simultaneously is of order c^2. It follows that the effect of triple interactions disappears, for a given duration of sedimentation, when the volume fraction c is sufficiently small, in which case the joint probability distribution of the position vectors of sphere centres remains approximately constant at its initial form.†

Information about the details of interactions between three neighbouring spheres will be needed before we can draw any firm conclusions about dispersions in which sedimentation has continued for an indefinite time. Four different parameters appear to be required for the specification of a three-sphere encounter (aside from the time variable), and so triple encounters are likely to be very complicated. It may happen that the effect of triple encounters on the probability distribution of the position of one sphere relative to another is weak, as a consequence of some cancellation of contributions from the many different types of triple encounter.

Another way of obtaining a hydrodynamically-determined steady state might be to calculate the probability density of the separation vector of two spheres of slightly different radii, a_1 and a_2 say, in a dispersion which contains these two sizes in some assumed proportion. The differential equation for the probability density obtained from a consideration of two-sphere encounters would then be of the form

$$\frac{\partial P(\mathbf{x}+\mathbf{r}|\mathbf{x})}{\partial t} = -(\mathbf{U}_1-\mathbf{U}_2).\nabla P - P\nabla.(\mathbf{U}_1-\mathbf{U}_2),$$

where $\mathbf{U}_1$ and $\mathbf{U}_2$ are the velocities of the two spheres when their centres are at $\mathbf{x}$ and $\mathbf{x}+\mathbf{r}$, and ∇ is here a differential operator with respect to $\mathbf{r}$. As $a_1-a_2 \to 0$, the rate of change of P tends to zero, but a definite equilibrium form for P presumably exists, however slow the rate of approach to it may be, and would be obtainable from this equation with the time derivative put equal to zero and with a_1-a_2 small but non-zero. (We see incidentally from (3.2) and (3.3) that, since $|\mathbf{W}(\mathbf{x},\mathbf{x}+\mathbf{r})|$ is of order a^4/r^4 when $a/r \ll 1$, both $\nabla.\mathbf{U}_1$ and $\nabla.\mathbf{U}_2$ are of order a^5/r^5 and so in the steady state $P(\mathbf{x}+\mathbf{r}|\mathbf{x})$ is equal to a constant (n) plus a term of order a^4/r^4, thus confirming – in this case – the claim made in §3 that $P(\mathbf{x}+\mathbf{r}|\mathbf{x})-P(\mathbf{x}+\mathbf{r})$ tends to zero 'rapidly' as $r/a \to \infty$.) In a particular experiment with a dispersion in which there were small unintended variations of size of

† This dependence of the probability distribution, and consequently of the mean sphere velocity, on the initial state of the dispersion is perhaps relevant to the interpretation of observations. If, contrary to what we have supposed, the dispersion is not 'well stirred' initially, there could be some variation of the observed mean sphere velocity, depending on chance features of the way in which the dispersion is prepared.

the spheres, the ultimate probability distribution might be determined by the slow changes due to these two-sphere encounters or by the changes due to the rare three-sphere encounters, depending on the circumstances.

Brownian motion might also have some influence in the case of a dispersion of very small spheres. The effect of Brownian motion on a sphere would be to tend to make it take all available positions in the fluid with equal probability, that is, to reduce the function $P(\mathbf{x}+\mathbf{r}|\mathbf{x})$ to the form (4.1).

Some observations concerning the spatial distribution of spheres have been made by Smith (1968). He used a dispersion of uniform acrylic spheres with volume fraction 0·025 in silicone liquid, the Reynolds number of the motion of one sphere being about 0·6. He found by actual counting that the frequency distribution of the number of sphere centres occurring within a spatial region of given volume was approximately of the binomial form, indicating that the probability of any particular sphere being located in the given region was statistically dependent on the positions of other spheres only inasmuch as they might diminish the available space in the given region. The chosen volume was of such a size as to contain 0·75 spheres on average, so that this is a fairly sensitive test of the statistical connexion between the positions of two neighbouring spheres.

It is evident that determination of the probability density $P(\mathbf{x}+\mathbf{r}|\mathbf{x})$ for a dispersion of ostensibly equal spheres and a long time of sedimentation presents a delicate problem, the answer to which may depend on the precise conditions. In most of the remainder of the paper we shall adopt the simple result (4.1) representing uniform probability for all physically accessible positions of one sphere relative to another. This well-defined and fundamental state is expected to apply to a dispersion of identical spheres within a certain time of sedimentation after some initial 'well stirred' state. In other circumstances a different form of the probability distribution may be found to be appropriate. The plan of calculation of $\overline{\mathbf{U}}$ described in the previous section can be carried out with any form for $P(\mathbf{x}+\mathbf{r}|\mathbf{x})$ which may be provided by later investigations. In §7 we shall consider one alternative form for $P(\mathbf{x}+\mathbf{r}|\mathbf{x})$.

With the adoption of the form (4.1) for the probability density of one sphere position relative to another, the bracketed expression occurring in (3.9) and (3.13) becomes

$$P(\mathbf{x}_0+\mathbf{r}|\mathbf{x}_0)-P(\mathbf{x}_0+\mathbf{r}) = \begin{cases} 0 & \text{if } r \geqslant 2a, \\ -n & \text{if } r < 2a. \end{cases} \tag{4.2}$$

Thus the mean velocity of a sphere is

$$\overline{\mathbf{U}} = \mathbf{U}_0+\overline{\mathbf{V}}'+\overline{\mathbf{V}}''+\overline{\mathbf{W}}$$

where

$$\overline{\mathbf{V}'} = -n\int_{r<2a} \mathbf{u}(\mathbf{x}_0, \mathbf{x}_0+\mathbf{r})\,d\mathbf{r} \tag{4.3}$$

$$\overline{\mathbf{V}''} = -n\int_{a<r<2a} \tfrac{1}{6}a^2\{\nabla^2\mathbf{u}(\mathbf{x}, \mathbf{x}_0+\mathbf{r})\}_{\mathbf{x}=\mathbf{x}_0}\,d\mathbf{r} + \tfrac{1}{2}c\mathbf{U}_0 \tag{4.4}$$

$$\overline{\mathbf{W}} = n\int_{r\geqslant 2a} \mathbf{W}(\mathbf{x}_0, \mathbf{x}_0+\mathbf{r})\,d\mathbf{r}, \tag{4.5}$$

in which $\mathbf{u}(\mathbf{x}_0, \mathbf{x}_0+\mathbf{r})$, $\{\nabla^2\mathbf{u}(\mathbf{x}, \mathbf{x}_0+\mathbf{r})\}_{\mathbf{x}=\mathbf{x}_0}$ and $\mathbf{W}(\mathbf{x}_0, \mathbf{x}_0+\mathbf{r})$ are to be interpreted as values of the functions $\mathbf{u}$, $\nabla^2\mathbf{u}$ and $\mathbf{W}$ at the point $\mathbf{x}_0$ as determined by the presence of a single sphere with centre at $\mathbf{x}_0+\mathbf{r}$. It will be recalled that $\mathbf{u}$ is the local velocity and that $\mathbf{W}(\mathbf{x}_0, \mathbf{x}_0+\mathbf{r})$ is given by (3.5).

For calculation purposes it is simpler to regard the centre of the 'presence' sphere as fixed, at the point $\mathbf{x}$ say, in which case the above integrals become

$$\overline{\mathbf{V}'} = -n\int_{r<2a} \mathbf{u}(\mathbf{x}+\mathbf{r}, \mathbf{x})\,d\mathbf{r} \tag{4.6}$$

$$\overline{\mathbf{V}''} = -n\int_{a<r<2a} \tfrac{1}{6}a^2\nabla_\mathbf{r}\mathbf{u}(\mathbf{x}+\mathbf{r}, \mathbf{x})\,d\mathbf{r} + \tfrac{1}{2}c\mathbf{U}_0 \tag{4.7}$$

$$\overline{\mathbf{W}} = n\int_{r\geqslant 2a} \mathbf{W}(\mathbf{x}+\mathbf{r}, \mathbf{x})\,d\mathbf{r}, \tag{4.8}$$

where

$$\mathbf{W}(\mathbf{x}+\mathbf{r}, \mathbf{x}) = \mathbf{U}(\mathbf{x}+\mathbf{r}, \mathbf{x}) - \mathbf{U}_0 - \mathbf{u}(\mathbf{x}+\mathbf{r}, \mathbf{x}) - \tfrac{1}{6}a^2\nabla_\mathbf{r}^2\mathbf{u}(\mathbf{x}+\mathbf{r}, \mathbf{x}). \tag{4.9}$$

5. Explicit calculation of the mean sphere velocity

The remaining part of the calculation of $\overline{\mathbf{U}}$ is a matter of detailed algebra.

We shall need to make use of the expression for the fluid velocity at $\mathbf{x}+\mathbf{r}$ due to a single rigid sphere of radius a with centre instantaneously at $\mathbf{x}$ falling through infinite fluid with velocity $\mathbf{U}_0$, viz.

$$\mathbf{u}(\mathbf{x}+\mathbf{r}, \mathbf{x}) = \mathbf{U}_0 \quad \text{for} \quad r \leqslant a$$

and
$$\mathbf{u}(\mathbf{x}+\mathbf{r}, \mathbf{x}) = \mathbf{U}_0\left(\frac{3a}{4r}+\frac{a^3}{4r^3}\right) + \mathbf{r}\frac{\mathbf{r}\,.\,\mathbf{U}_0}{r^2}\left(\frac{3a}{4r}-\frac{3a^3}{4r^3}\right) \quad \text{for} \quad r \geqslant a. \tag{5.1}$$

The corresponding expression for $\nabla_\mathbf{r}^2\mathbf{u}$ at points in the fluid is

$$\nabla_\mathbf{r}^2\mathbf{u} = \mathbf{U}_0\frac{3a}{2r^3} - \mathbf{r}\frac{\mathbf{r}\,.\,\mathbf{U}_0}{r^2}\frac{9a}{2r^3}. \tag{5.2}$$

It follows that

$$\int_{r\leqslant a} \mathbf{u}(\mathbf{x}+\mathbf{r}, \mathbf{x})\,d\mathbf{r} = \tfrac{4}{3}\pi a^3\mathbf{U}_0, \quad \int_{a<r<2a} \mathbf{u}(\mathbf{x}+\mathbf{r}, \mathbf{x})\,d\mathbf{r} = 6\pi a^3\mathbf{U}_0,$$

and then from (4.6) we have

$$\overline{\mathbf{V}'} = -\tfrac{22}{3}\pi a^3 n\mathbf{U}_0 = -\tfrac{11}{2}c\mathbf{U}_0. \tag{5.3}$$

Similarly we find

$$\int_{a<r<2a} \tfrac{1}{6}a^2\nabla_\mathbf{r}^2\mathbf{u}(\mathbf{x}+\mathbf{r}, \mathbf{x})\,d\mathbf{r} = 0,$$

whence, from (4.7),

$$\overline{\mathbf{V}''} = \tfrac{1}{2}c\mathbf{U}_0. \tag{5.4}$$

Consider now the mean value of $\mathbf{W}(\mathbf{x}+\mathbf{r}, \mathbf{x})$ which in view of (4.9), (5.1) and (5.2) can be written as

$$\mathbf{W}(\mathbf{x}+\mathbf{r}, \mathbf{x}) = \mathbf{U}(\mathbf{x}+\mathbf{r}, \mathbf{x}) - \mathbf{U}_0 - \mathbf{U}_0\left(\frac{3a}{4r}+\frac{a^3}{2r^3}\right) - \mathbf{r}\frac{\mathbf{r}\,.\,\mathbf{U}_0}{r^2}\left(\frac{3a}{4r}-\frac{3a^3}{2r^3}\right). \tag{5.5}$$

In this relation $\mathbf{U}(\mathbf{x}+\mathbf{r}, \mathbf{x})$ is the exact translational velocity of each one of a pair of rigid spheres in infinite fluid with centres separated by the vector $\mathbf{r}$, and

$\mathbf{W}(\mathbf{x}+\mathbf{r}, \mathbf{x})$ can be regarded as the difference between this exact value and the approximation to $\mathbf{U}$ obtained by modifying the stress distribution at the surface of each sphere to take account of the velocity distribution due to the other as if that other sphere were alone in infinite fluid. As mentioned in § 3, the magnitude of $\mathbf{W}(\mathbf{x}+\mathbf{r}, \mathbf{x})$ behaves as $U_0 a^4/r^4$ when $a/r \ll 1$.

The translational velocity of a pair of identical spheres falling in infinite fluid has been found in closed form by Stimson & Jeffrey (1926) for the case in which the line of centres is vertical and by Goldman, Cox & Brenner (1966) for a horizontal line of centres (with the two spheres free to rotate). The problem is linear, and so there are only two independent cases. We may therefore write the common translational velocity of two identical rigid spheres with centres separated by the vector $\mathbf{r}$ as

$$\mathbf{U}(\mathbf{x}+\mathbf{r},\mathbf{x}) = \lambda_1 \mathbf{r}\frac{\mathbf{r}\,.\,\mathbf{U}_0}{r^2} + \lambda_2\left(\mathbf{U}_0 - \mathbf{r}\frac{\mathbf{r}\,.\,\mathbf{U}_0}{r^2}\right), \tag{5.6}$$

where $\mathbf{U}_0$ is the velocity with which either sphere would fall in isolation. λ_1 and λ_2 are inverse resistance coefficients for vertical and horizontal line of centres respectively which are functions of the separation distance r alone and which are

$\dfrac{r}{a}$	λ_1	λ_2	$\left\{\lambda_1+2\lambda_2-3\left(1+\dfrac{r}{a}\right)\right\}\dfrac{r^2}{a^2}$
2·0	1·5500	1·3799	−0·761
2·0049	1·5494	1·4027	−0·569
2·0907	1·5376	1·3933	−0·484
2·2553	1·5160	1·3648	−0·431
2·6749	1·4662	1·3029	−0·355
3·0862	1·4236	1·2586	−0·298
4·0	1·3472	1·1950	−0·203
6·0	1·2427	1·1273	−0·100
8·0	1·1847	1·0947	−0·058

TABLE 1. Values of the inverse resistance coefficients for vertical (λ_1) and horizontal (λ_2) line of centres of an isolated pair of equal spheres with centres distance r apart

known from the two papers just mentioned. The integral of $\mathbf{W}(\mathbf{x}+\mathbf{r},\mathbf{x})$ over a spherical surface of radius r with centre at $\mathbf{x}$ is then found from (5.5) and (5.6) to be

$$4\pi r^2 \mathbf{U}_0\{\tfrac{1}{3}(\lambda_1+2\lambda_2)-1-a/r\},$$

and the average value of $\overline{\mathbf{W}}$ follows from (4.8) as

$$\overline{\mathbf{W}} = c\mathbf{U}_0\int_2^\infty \left\{\lambda_1+2\lambda_2-3\left(1+\frac{a}{r}\right)\right\}\frac{r^2}{a^2}d\left(\frac{r}{a}\right). \tag{5.7}$$

Table 1 shows values of λ_1 and λ_2 for some values of r/a between 2 and 8 obtained from the reciprocals of these quantities given by Goldman *et al.* 1966 (who obligingly provided numerical values of λ_1^{-1} from Stimson & Jeffrey's solution along with their own calculations of λ_2^{-1}). Table 1 also gives values of the integrand of (5.7) over this range; for larger values of r/a, the integrand is given sufficiently accurately by its asymptotic form, which is readily found by the method described in §3 to be $-\frac{15}{4}(a/r)^2$. By numerical integration† over the range $2 \leqslant r/a \leqslant 8$ (using more data than is given in table 1) and algebraic integration of the asymptotic form over the range $r/a \geqslant 8$ it was found that

$$\overline{\mathbf{W}} = -1{\cdot}55c\mathbf{U}_0, \tag{5.8}$$

correct to the second decimal place in the numerical coefficient.

The results (5.3), (5.4) and (5·8), together with (3.14), show that the mean translational velocity of a sphere in a dispersion of identical rigid spheres with (small) volume concentration c is

$$\begin{aligned} \overline{\mathbf{U}} &= \mathbf{U}_0 + c\mathbf{U}_0(-5{\cdot}50 + 0{\cdot}50 - 1{\cdot}55) \\ &= \mathbf{U}_0(1 - 6{\cdot}55c), \end{aligned} \tag{5.9}$$

correct to order c.

Now that the resultant change in the mean settling speed due to particle interactions has been determined, it is instructive to look back and compare the physical meanings and magnitudes of the various contributions. The downward flux of volume of solid material in the dispersion is accompanied, in a homogeneous dispersion with zero mean volume flux at each point, by a corresponding net upward flux of fluid volume; this change in the fluid environment for one sphere causes the mean settling speed to differ from the value it would have in infinite clear fluid by an amount $-c\mathbf{U}_0$ (correct to the order c). The falling spheres also drag down with them some adjoining fluid, and this downward flux of fluid volume in the inaccessible shells surrounding the rigid spheres is accompanied by an equal upward flux of volume in the part of the fluid that *is* accessible to the centre of a test sphere, the corresponding contribution to the change in mean settling velocity being $-\frac{9}{2}c\mathbf{U}_0$. (These two contributions associated with conservation of volume flux together comprise the term denoted by $\overline{\mathbf{V}'}$.) The motion of the spheres generates collectively a velocity distribution in the fluid such that the second derivative of the velocity (or $\nabla^2\mathbf{u}$, more precisely) has non-zero mean, and this property of the environment for a particular sphere changes its mean velocity by an amount $+\frac{1}{2}c\mathbf{U}_0$. Finally, when the test sphere whose velocity is being averaged is near one of the other spheres in the dispersion, the interaction between these two spheres gives the test sphere a translational velocity which is significantly different from that which is estimated from the velocity distribution in the fluid in the absence of the test sphere, and so the previous three contributions need to be supplemented; this gives a further change in the mean settling speed equal to $-1{\cdot}55c\mathbf{U}_0$.

The largest single contribution to the change in the settling speed thus comes from the diffuse upward current which compensates for the downward flux of fluid volume in the inaccessible shells surrounding the rigid spheres; and this contribution, like the net change in the settling speed, is a reduction in the settling speed.

† I am grateful to Mr E. J. Hinch for carrying out these computer calculations for me.

6. Comparison with other work

As noted in §1, there are two previous investigations (Burgers 1942; Pyun & Fixman 1964) which have produced the result that $\bar{\mathbf{U}}-\mathbf{U}_0$ is proportional to c for a dilute dispersion. Both investigations assumed the spheres to be randomly located, with equal probability for all physically possible configurations (although Burgers also considered a regular array), both employed probability methods, and both recognized that there are convergence difficulties when the effects of adjoining spheres on a test sphere are taken separately and summed over all the spheres in the dispersion. However, the statistical methods employed were *ad hoc* in character rather than systematic, and neither set of authors overcame satisfactorily the convergence difficulties. Burgers adopted an expression for the velocity of a sphere which was effectively that given by (3.3) with (3·2). He then calculated the average value of the term $\mathbf{V}'(\mathbf{x}_0,\mathscr{C}_N)$ from a consideration of the flow field due to a single falling sphere, essentially by the device of introducing a certain distributed body force on the fluid which makes the net volume flux across any infinite horizontal plane identically zero and which turns the r^{-1} asymptotic behaviour of the velocity due to the falling sphere into behaviour as r^{-3}; and he expressed the mean value of $\mathbf{V}''(\mathbf{x}_0,\mathscr{C}_N)$ as a volume integral over the possible positions of a single falling sphere which was not absolutely convergent but which he made determinate by carrying out the integration in three dimensions in a particular way, the result (which is correct!) being accepted with obvious unease. Pyun & Fixman dealt correctly with the reduction of $\overline{\mathbf{V}'}$ to an absolutely convergent integral involving the velocity distribution of a single falling sphere, essentially by use of the exact volume flux relation (3.7) as in this paper (although they describe this device curiously as a change of reference frame); but they failed to notice that the volume integral of

$$\mathbf{U}(\mathbf{x}_0,\mathbf{x}_0+\mathbf{r})-\mathbf{U}_0-\mathbf{V}'(\mathbf{x}_0,\mathbf{x}_0+\mathbf{r})$$

over all possible positions of the sphere with centre at $\mathbf{x}_0+\mathbf{r}$ is still not absolutely convergent, and, although as a consequence of continuing to overlook the term $\frac{1}{6}a^2\nabla^2\mathbf{u}$ in the expression for the velocity of a sphere due to a non-uniform environment their explicit integrals were absolutely convergent, the outcome was that the contribution $\overline{\mathbf{V}''}$ was missing from their calculation. Finally neither Burgers nor Pyun & Fixman had access to the results of the recent exact calculation by Goldman, Cox & Brenner (1966) of the speed of fall of a pair of identical spheres with horizontal line of centres, and so they were able to make only an approximate calculation of $\overline{\mathbf{W}}$ using the method of reflexions; Burgers used only the asymptotic form for the velocity of one sphere at distance r from a second sphere and got $\overline{\mathbf{W}} = -1{\cdot}88c\mathbf{U}_0$, whereas Pyun & Fixman took more terms in the sequence of reflexions and obtained $\overline{\mathbf{W}} = -1{\cdot}66c\mathbf{U}_0$, the accurate value from §5 being $-1{\cdot}55c\mathbf{U}_0$. The overall result for $\bar{\mathbf{U}}-\mathbf{U}_0$ from Burgers's calculation for a random distribution of spheres was $-6{\cdot}88c\mathbf{U}_0$, and that of Pyun & Fixman was $-7{\cdot}16c\mathbf{U}_0$. Neither result is far from the value obtained here ($-6{\cdot}55c\mathbf{U}_0$). The intended contribution of the present paper is not so much a new numerical value of $\bar{\mathbf{U}}$ as the provision of a systematic method of solution which can be trusted and which may be applicable to a range of problems involving randomly dispersed particles.

Measurement of the average velocity of a particular sphere in a dispersion over a long time is difficult, and many observers have sought other kinds of average of the particle velocity, such as the speed of fall of the relatively sharp 'top' to the cloud of particles in a vessel. The available data of different kinds have been examined by Maude & Whitmore (1958) who concluded that the observed mean settling velocity of identical spheres moving with small Reynolds number is best represented, over a wide range of values of c including small values, by a relation of the form

$$\overline{\mathbf{U}} = \mathbf{U}_0(1-c)^{\beta},$$

in which the value of the constant β 'is uncertain but is approximately 5'. The agreement between this empirical relation in the case $c \ll 1$ and the theoretical result (5.9) is not worse than one would expect from the general scatter of the observations. There is a need for more decisive experiments on the average falling speed of spheres specifically for small values of c (say, c less than about 0·05).

There is a persistent belief in the literature that the decrease in the settling speed in a dilute suspension should be proportional to $c^{\frac{1}{3}}$, despite the fact that such a relation predicts appreciably larger decreases in settling speed than are observed The basis of this belief is presumably that all the calculations for a regular array of spheres, and all those using a cell model, give a result of this form.† It is worthwhile to consider in general terms the reasons why these methods yield a result which is different in form from that found in this paper.

We may note first that any method of calculation which represents each falling sphere wholly as a point force $\mathbf{G}$ applied to the fluid at the position of the sphere centre is certain on dimensional grounds to find that the additional velocity near one sphere due to the presence of all the others is proportional to $\mathbf{G}/\mu R$, because in this linear problem the external force $\mathbf{G}$ acting on each sphere must appear linearly on the right-hand side and some length R associated with the particle arrangement is the only length entering the problem. The force $\mathbf{G}$ can be written as $6\pi a\mu\mathbf{U}_0$, and so the change in fluid velocity at the position of a particular sphere due to the presence of all the other spheres – which is equivalent to the change in fall velocity of the chosen sphere to the first order in c, in these calculations – would be predicted to be of the form

$$\mathbf{U}-\mathbf{U}_0 \propto a\mathbf{U}_0/R.$$

In the case of a regular geometrical arrangement of the spheres, R can be taken as the distance between neighbouring spheres, whence a/R is proportional to $c^{\frac{1}{3}}$. Likewise in the case of a cell model, R can be taken only as a linear dimension of the cell and then, with the same volume fraction of solid material in the cell as in the real dispersion, a/R is proportional to $c^{\frac{1}{3}}$. In all the published investigations

† The result of a numerical calculation for a semi-random distribution of spheres by Famularo & Happel (1965) is also quoted in this form, but this investigation was for only one value of the concentration and does not provide evidence concerning the functional dependence on c.

which adopt either a regular arrangement of the spheres or a cell model, it appears that, so far as the derivation of the first approximation to $\mathbf{U}-\mathbf{U}_0$ for $c \ll 1$ is concerned, each sphere was represented by a point force and the radius a entered the analysis only through either the identity connecting $\mathbf{G}$ and $\mathbf{U}_0$ or the choice of R as a function of c. A result of the form

$$\mathbf{U}-\mathbf{U}_0 \propto c^{\frac{1}{3}}\mathbf{U}_0$$

is consequently inevitable.

We can also see more directly how this result comes about and why it differs from the result for a random distribution of spheres. The key point of the explanation is that, since the fluid velocity due to a single moving sphere varies with distance r as $G/\mu r$ (or $U_0 a/r$), any method which selects and takes account only of one particular value (or a small number of comparable values or multiples) of the distance between spheres, the average spacing R being the inevitable choice, as do both the regular-array method and the cell-method, will find a resultant induced velocity at the position of one sphere which is of order $U_0 a/R$. On the other hand, for a random distribution of spheres such that the position vector of the centre of a sphere takes all accessible values with equal probability, there is no direct significance or preference attached to the average particle separation. The result for a random distribution of spheres is different, because a test sphere samples all accessible positions in the fluid and the change in fall speed due to the presence of the other spheres is determined primarily by the fact that, since the average velocity over a large volume of the dispersion is zero, the integral of the fluid velocity over positions accessible to the test sphere is minus the integral over the inaccessible region, which occupies a fraction of the total volume of order c and in which the velocity is of order U_0.

7. A different form for the probability distribution of the separation of two spheres

In view of the conclusion in §4 that there is likely to be some variation of the probability density function $P(\mathbf{x}+\mathbf{r}|\mathbf{x})$ under different practical conditions, it is desirable to consider the sensitivity of the mean settling velocity to the assumption made about this function. The formulae given in §§3, 5 allow the mean settling speed to be calculated without difficulty for any given form for $P(\mathbf{x}+\mathbf{r}|\mathbf{x})$, and, as an example of a form different from that represented by (4.1), we shall suppose that there is an excess number of pairs of spheres (additional to the number corresponding to uniform probability density over all accessible positions) with separations close to the minimum value $2a$. The excess number of nearly-touching pairs can be represented conveniently as a fraction of c, and we shall suppose that this excess number, αc say, is spread symmetrically over all directions of $\mathbf{r}$. The analytical form of the new probability density function for the separation vector of two spheres thus differs from (4.1) by the addition of a term

$$\frac{\alpha c}{16\pi a^2}\delta(r-2a) \tag{7.1}$$

to the right-hand side, where $\delta(r-2a)$ is the delta function defined by

$$\delta(s) = 0 \text{ when } s \neq 0, \quad \int_{-\infty}^{\infty} \delta(s)\, ds = 1.$$

The resulting addition to $\bar{\mathbf{U}}$ is then

$$\frac{\alpha c}{16\pi a^2} \int_{r=2a} \{\mathbf{u}(\mathbf{x}+\mathbf{r},\mathbf{x}) + \tfrac{1}{6}a^2\nabla_{\mathbf{r}}^2\mathbf{u}(\mathbf{x}+\mathbf{r},\mathbf{x}) + \mathbf{W}(\mathbf{x}+\mathbf{r},\mathbf{x})\}\, d\mathbf{r}, \tag{7.2}$$

which we see from the formulae (5.1), (5.2), (5.5) and (5.6) to be

$$\begin{aligned} &\alpha c\mathbf{U}_0\left[\tfrac{1}{2} + \tfrac{1}{3}\left\{\lambda_1 + 2\lambda_2 - 3\left(1+\frac{r}{a}\right)\right\}_{r=2a}\right] \\ &= \alpha c\mathbf{U}_0(0{\cdot}50 - 0{\cdot}06) = 0{\cdot}44\, \alpha c\mathbf{U}_0. \end{aligned} \tag{7.3}$$

Table 1 shows that λ_2 varies rapidly with r near $r = 2a$. However, it appears that the change in $\bar{\mathbf{W}}$ is small compared with that in $\overline{\mathbf{V}'}$ (which is represented by the 0·5 in (7.3)) in any event, and so the change in $\bar{\mathbf{U}}$ is approximately the same when the excess pairs of spheres are in contact as when they are only nearly touching.

As was to be expected, a positive value of α, corresponding to a larger number of close pairs of spheres than that assumed in the previous calculation, leads to an increase in the mean settling speed. The value $\alpha = 1$ corresponds to an excess average number of sphere centres at distance $2a$ from the centre of a given sphere which is equal to the average number of sphere centres in a volume $\frac{4}{3}\pi a^3$; and for $\alpha = 1$ we have

$$\bar{\mathbf{U}} - \mathbf{U}_0 = c\mathbf{U}_0(-6{\cdot}55 + 0{\cdot}44).$$

When $\alpha = 15$, corresponding to an excess number of nearly-touching spheres equal to the average number of sphere centres in 15 times the volume of a sphere, the addition to the mean settling speed due to these close pairs is just sufficient to cancel the net hindering effect of the spheres with uniform probability density over all accessible positions. Any attractive force between neighbouring spheres which tends to bring together pairs of spheres initially a few radii apart (which would of course produce a deficiency of sphere centres at distances of several radii from a given sphere, but this has a smaller effect on $\bar{\mathbf{U}}$ than the excess number of close pairs) is thus likely to have an appreciable effect on the value of $\bar{\mathbf{U}} - \mathbf{U}_0$ and perhaps to make the net change in the average speed of fall an increase. A repulsive force between neighbouring spheres on the other hand would lead to a probability distribution for which $\alpha < 0$ and so to an even greater reduction in the average speed of fall.

Appendix. The translational velocity of a rigid sphere placed in fluid with a given velocity distribution

We suppose that the velocity distribution in a body of fluid, before the introduction of a rigid sphere, is $\mathbf{u}(\mathbf{x})$. If a surface distribution of force of density $\mathbf{F}$ is now applied to the fluid at a spherical surface A_0 of radius a centred on $\mathbf{x}_0$, the velocity distribution will be changed. In the absence of any boundaries to the fluid, the additional velocity at point $\mathbf{x}$ due to this surface distribution of force is

$$\frac{1}{8\pi\mu}\int_{A_0} I_{ij}(\mathbf{x}-\mathbf{x}')\,F_j(\mathbf{x}')\,dA(\mathbf{x}'), \tag{A 1}$$

where
$$I_{ij}(\mathbf{x}) = \frac{\delta_{ij}}{|\mathbf{x}|}+\frac{x_i x_j}{|\mathbf{x}|^3}.$$

If this surface distribution of force is to represent the presence of a rigid sphere with surface A_0, the total velocity on A_0 must have the form

$$\mathbf{U}+\mathbf{\Omega}\times(\mathbf{x}-\mathbf{x}_0)$$

corresponding to a translational velocity $\mathbf{U}$ and angular velocity $\mathbf{\Omega}$ of the sphere. Hence the integral equation determining $\mathbf{F}$ is

$$u_i(\mathbf{x})+\frac{1}{8\pi\mu}\int_{A_0} I_{ij}(\mathbf{x}-\mathbf{x}')\,F_j(\mathbf{x}')\,dA(\mathbf{x}') = U_i+\{\mathbf{\Omega}\times(\mathbf{x}-\mathbf{x}_0)\}_i, \tag{A 2}$$

to be satisfied at each point of A_0.

The desired result is now obtained by integrating this relation over the surface A_0. It can be shown by straight-forward integration that

$$\int_{A_0} I_{ij}(\mathbf{x}-\mathbf{x}')\,dA(\mathbf{x}) = \tfrac{16}{3}\pi a\delta_{ij}$$

when $\mathbf{x}'$ lies on A_0, whence we find

$$\mathbf{U} = \frac{1}{4\pi a^2}\int_{A_0}\mathbf{u}(\mathbf{x})\,dA(\mathbf{x})+\frac{1}{6\pi a\mu}\int_{A_0}\mathbf{F}(\mathbf{x})\,dA(\mathbf{x}). \tag{A 3}$$

The last term is proportional to the total excess weight of the sphere, and can be represented in terms of the velocity $\mathbf{U}_0$ which this total weight would give the sphere in infinite fluid otherwise at rest, giving

$$\mathbf{U} = \mathbf{U}_0+\frac{1}{4\pi a^2}\int_{A_0}\mathbf{u}(\mathbf{x})\,dA(\mathbf{x}). \tag{A 4}$$

As noted above, this expression for the translational velocity of the inserted rigid sphere is applicable only in the absence of boundaries to the fluid. If boundaries exist at which certain conditions must be satisfied, the additional velocity resulting from the introduction of a surface distribution of force on A_0 is the sum of the expression (A 1) and a contribution from a certain system of image forces.

The relation (A 4) is equivalent to the first of Faxen's laws (see Oseen 1927). For the idea underlying this simple proof I am indebted to Dr J. R. Smith, of Mount Allison University, New Brunswick, who observed that the relation (A 3) with $\mathbf{u} = 0$ provides a neat derivation of Stokes's law for the resistance to a moving sphere in unbounded fluid.

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